Mixed Observable Estimation of Random Thrust Errors for Solar Electric Propulsion Spacecraft

Robert B. Asher*
ORINCON Corp., La Jolla, Calif.

Thomas J. Eller†
US Air Force Academy, Colo.

Stanley R. Robinson‡
Air Force Institute of Tech., Wright-Patterson AFB, Ohio

Mark D.A. Shackelford*
Columbus Air Force Base, Mass.

Byron D. Tapley¶
University of Texas at Austin, Austin, Tex.

This paper considers the development of an estimation theory for mixed measurement sets. The measurement sets include both doubly stochastic Poisson process observables and additive noise observables. The estimation theory is presented in detail. The mixed observable estimation theory is applied to the problem of estimating the random thrust errors of a solar electric vehicle, where the effective thrust vector produced by the electric propulsion system is not equal to the desired thrust because of angle and magnitude errors in thrust, due to attitude deadband control errors and other error random sources. The random errors are modeled by appropriate models. The measurements consist of a Doppler measurement of the additive white noise type, and two photon counting star-tracker measurements with realistic noise fundamental limits associated with the hardware implementation.

I. Introduction

THE solar electric propulsion spacecraft is driven by an electric engine which obtains its power from solar energy conversion devices. The effective thrust vector produced by the electric propulsion system is not equal to the desired thrust for a variety of reasons. The angle and magnitude errors in thrust are due to attitude deadband control errors; other error sources are random in nature. Eller[1] and Tapley and Hagar[2,3] treat the problem of estimating the random thrust given Doppler and star-tracker measurements. Both measurements are assumed to be of the additive white noise type. It is the intent of this paper to treat the estimation of the random thrusts by again using a Doppler measurement of the additive white noise type, but with two star-tracker measurements with realistic noise fundamental limits associated with the hardware implementation. In particular, the fundamental limits are associated with the photon counting nature of the star tracker used. The measurement set is, therefore, a mixed observable of both an additive white noise measurement and a doubly stochastic Poisson process measurement. Estimation theory of this type has been studied elegantly using the Martingale theory by Vaca and Snyder.[4,5] The estimation equations for this type of measurement set are derived independently in a much simpler and more easily understood method. In particular, the differential equation for the evolution of the probability density function conditioned on the additive noise and the doubly stochastic Poisson process measurements is derived similarly to Kushner.[7] The adaptive estimation equations, which are not treated by Vaca and Snyder, are derived by use of the smoothing property of expectations in a manner similar to that of Asher and Lainiotis.[12] Moment approximations to the exact conditional mean estimator are used to obtain an extended form of the estimator for implementation. These equations are applied to the problem of estimating the random thrust errors of the solar electric vehicle, with two star trackers yielding photon count measurements and the Doppler white noise measurement. The bandwidth of the random thrust errors is assumed uncertain, and the adaptive estimation scheme derived as well.

The modeling of the star trackers includes a photon counting process based upon a quad cell photon counter. The star tracker works off a null position and continually monitors the photon imbalance on the quad cell due to the Airy disk shifting associated with the vehicle attitude changing relative to the star. This yields a doubly stochastic Poisson process in which the vehicle attitude changes are assumed to cause a correlated disturbance in the angle of arrival relative to the vehicle. This is coupled into the thrust errors and the thrust error states modulate the intensity rate of the Poisson counting processes. The complete theory is derived and estimator equations set up.

Investigators have concluded that stochastic nongravitational accelerations resulting from random variations in the thrust vector are the dominant error source for

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*Sr. Principal Engineer. Member AIAA.

†Assistant Dean. Member AIAA.

‡Assistant Professor, Dept. of Electrical Engineering.

§Formerly Cadet, U.S. Air Force Academy, Colo. Student Member AIAA.

¶Professor, Dept. of Aerospace Engineering and Engineering Mechanics. Member AIAA.

navigation of solar electric propulsion spacecraft (SEP) interplanetary missions.[10] The SEP spacecraft obtains its power from solar energy conversion devices. The effective thrust vector produced by the SEP system is not equal to the desired thrust for a variety of reasons. The angle and magnitude errors in thrust are random in nature and are due to a variety of sources. For the purpose of this study, the errors are assumed to be due to attitude control deadband errors.

If, on the basis of measurements, these random thrust errors can be estimated accurately, then the entire state vector of the spacecraft can be estimated and its orbit successfully determined.

References 1-3 treat the problem of estimating the SEP thrust errors through the use of dynamic model compensation techniques,[11] given conventional Doppler and star-tracker measurements that are assumed to be of the additive white noise type. The same basic approach is used here with the exception of the assumed mechanization of the star tracker.

The star trackers used in this paper are assumed to be in a photon counting mode. The photon counting mode yields the fundamental limits of accuracy for the star tracker. Since the basic measurements from the star trackers are quantum photoevents, it is necessary to model the basic physical elements of the measurements. Reference 9 gives detailed results on photon counting statistics. Basically, the process may be described as a doubly stochastic Poisson process. That is, given the intensity rate, the process is Poisson with the probability of n counts in the time interval $[0,t]$ as

$$P_r\{N_{0,t}=n\,|\lambda(t)\}=\left[\left(\int_0^t \lambda d\sigma\right)^n \exp\left(-\int_0^t \lambda d\sigma\right)\right]\Big/ n!$$

However, the intensity rate λ is a function of a stochastic process, i.e., $\lambda=\lambda\{x(t)\}$ where $x(t)$ is a stochastic process. It is possible to estimate the stochastic process $x(t)$ from measurements of the photoevents. This theory is called estimation of doubly stochastic Poisson processes.

In this paper, it will be shown that the intensity rate models for the star trackers are functions of the thrust errors. Thus, the photoevents on the detectors are measurements of these stochastic processes. Therefore, this part of the process is a doubly stochastic Poisson process. However, because the range-rate measurements are of the additive white noise type, the combination of this measurement with photon counting gives us a mixed observable estimation process. The mixed observable problem is that of using both doubly stochastic Poisson process measurements and additive white noise measurements together in the same process. Thus, we have the problem of estimating the position, velocity, thrust error, and auxiliary states on the basis of mixed observables. The theory for this type of estimation has been solved independently from that of Refs. 4-6, and is a much simpler proof. The theory is given for this type of process and is applied to the optimal estimation of the solar electric propulsion spacecraft states.

The paper is divided into five sections. Section II gives a detailed problem statement. Section III contains the development of the state and measurement equations to be used. Section IV contains the estimator equations, and Sec. V contains concluding remarks. The appendices contain the development of the estimation equations and the calculation of partial derivatives of the observation-state relationships.

II. Problem Statement

The scenario used to illustrate the application of estimation based on mixed observables is the problem treated in detail in Refs. 1-3. The problem is to estimate the trajectory of a continuous low-thrust, solar-electric propulsion spacecraft subject to time-correlated errors in thrust acceleration. The nominal 5-month mission is from heliocentric injection near the Earth's sphere of influence to a fly-by of the Asteroid Eros at 1.45 a.u.

If we assume that the spacecraft motion is due solely to the two-body central force attraction of solar gravity and the thrust acceleration of the body-fixed propulsion system, then the vector equation of translational motion of the point mass vehicle is

$$\ddot{r}=-(\mu/|r|^3)r+T \tag{1}$$

where r is a 3-vector of heliocentric-ecliptic position components X, Y, Z; $\ddot{r}$ is the second derivative of r with respect to time; μ is the gravitational parameter of the sun; $|r|$ is the magnitude of r; and T is the heliocentric thrust vector. T is composed of both the nominal thrust acceleration T^* and thrust acceleration errors from a number of sources, but principally from vehicle attitude errors. The actual thrust acceleration vector is referenced to a nominal thrust vector which we assume to be of constant magnitude and along the y axis of the orbital coordinate frame. The actual thrust vector may be expressed in the various frames of reference shown in Figs. 1 and 2.

In the heliocentric frame

$$T=\begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}=\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix} \tag{2}$$

where ψ is heliocentric right ascension. In the orbital frame,

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}=a\begin{bmatrix} \sin\gamma\cos\theta \\ \cos\gamma \\ \sin\gamma\sin\theta \end{bmatrix} \tag{3}$$

where

$$a=a^*+\delta a \tag{4}$$

where a^* is the magnitude of the nominal thrust acceleration and δa is the error in magnitude. θ and γ are clock and cone angles that represent vehicle attitude errors which are the source of the random thrust direction errors.

Coordinates x, y, and z express the orbital coordinate frame oriented such that x and y are parallel to the ecliptic, z is along Z, and y is perpendicular to the heliocentric position vector of the vehicle. The thrust vector is assumed to be fixed in a vehicle coordinate frame x', y', z' nominally aligned with the orbital frame.

Observations

Range Rate

Range rate, the relative motion of the vehicle along the line of sight from the tracking station to the vehicle, may be expressed in equation form as

$$\dot{\rho}=(\dot{\bar{\rho}}\cdot\bar{\rho})/\rho \tag{5}$$

where $\bar{\rho}$ and $\dot{\bar{\rho}}$ are the position and velocity, respectively, of the vehicle relative to the tracking station, and the scalar ρ is the magnitude of $\bar{\rho}$. The motion of the tracking station is due to both rotation and orbital revolution of the Earth. The range-rate observation is assumed to consist of a deterministic value plus zero-mean, normally distributed random error ν with statistics $E[\nu_i]=0$ and $E[\nu_i\nu_j]=\sigma_i^2\delta_{ij}$, where δ is the Kronecker delta.

Star Trackers

Vehicle attitude deviations from the nominal orientation cause the body-fixed photon counting star trackers to sense

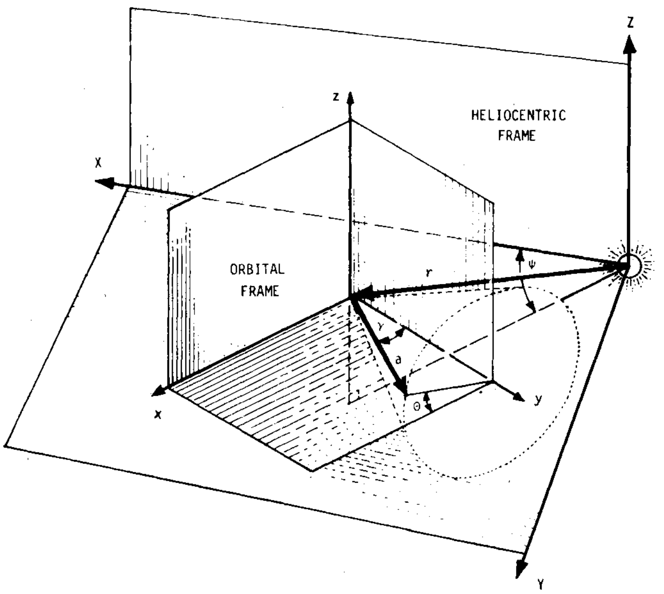

Fig. 1 Reference frames.

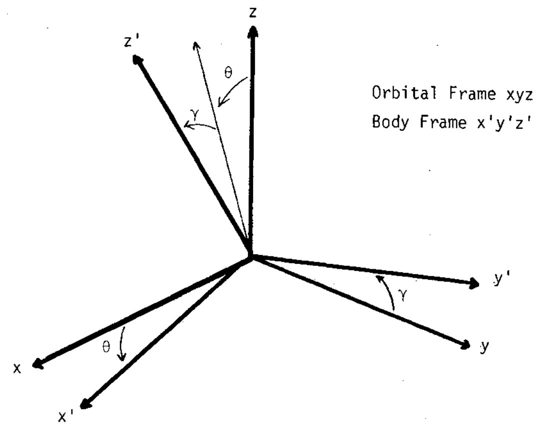

Fig. 2a Body frames, orbital.

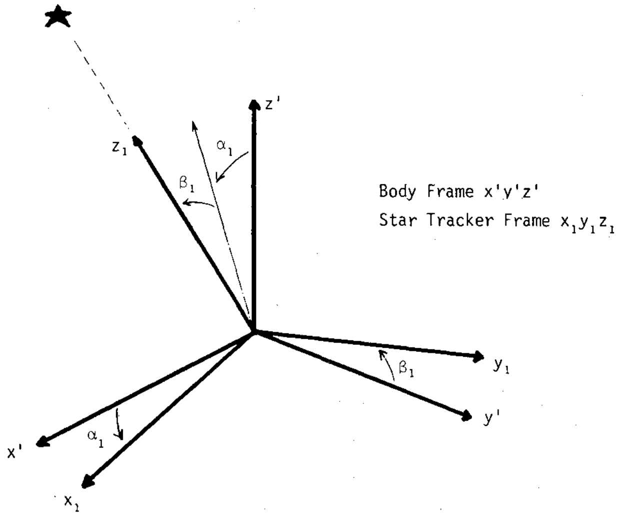

Fig. 2b Star-tracker frames, body.

star direction errors. Since both the thrust vector and star trackers are assumed fixed relative to the vehicle, the star trackers directly measure the thrust vector direction errors. As the vehicle rotates, each quadrant of the star tracker will yield a photoevent count proportional to the angle of rotation. It is tacitly assumed that all rotations are small. Figure 3 shows the optical schematic of a star tracker. Figure 2 shows star tracker orientation relative to the body coordinate frame.

Since the star trackers yield count measurements which are Poisson processes given the intensity rate, the estimation theory must be based on the doubly stochastic Poisson process nature of the measurements. Furthermore, since the measurements are mixed in that they include the Doppler measurements of the additive noise type, it is necessary to derive an estimation theory that includes both types of measurements. In this problem, it is necessary to estimate the state of the process.

$$\mathrm{d}x(t) = f[x,(t),t]\,\mathrm{d}t + G(t)\,\mathrm{d}\beta(t) \tag{6}$$

where x is an n vector and $\mathrm{d}\beta$ is an m vector of independent Wiener processes with $E\{\mathrm{d}\beta(t)\mathrm{d}\beta^T(t)\} = Q(t)\mathrm{d}t$ given the measurements

$$\mathrm{d}y(t) = h[x(t),t]\,\mathrm{d}t + \mathrm{d}\eta(t) \tag{7}$$

where y is a q vector, $\mathrm{d}\eta$ is a vector of independent Wiener processes with $E\{\mathrm{d}\eta(t)\mathrm{d}\eta^T(t)\} = R(t)\mathrm{d}t$, and measurements consisting of the sequence of q_λ vector of counts $N_t = \{N(\sigma), t_0 \le \sigma \le t\}$ with each element N_i obtained from a doubly stochastic Poisson process

$$P_r\{N_{i_t} - N_{i_s} = n \mid \lambda[x(\sigma)], \qquad s<\sigma\le t\}$$
$$= (n!)^{-1}\left(\int_s^t \lambda_i[x(\sigma)]\,\mathrm{d}\sigma\right)^n \exp\left(-\int_s^t \lambda_i[x(\sigma)]\,\mathrm{d}\sigma\right) \tag{8}$$

where $N_{i_t} - N_{i_s}$ is the total number of counts for the ith element in the interval $[s,t]$. The extended estimation equations for this process are given in Sec. IV and are developed in Appendix A.

III. Filter State and Measurement Equations

In accordance with the Dynamic Model Compensation (DMC) technique,[11] we assume that the thrust acceleration error, and thus by inference, vehicle attitude error, can be separated into modeled and error components $T = T + m(t)$, where $m(t)$ is a 3-vector of thrust acceleration error components.[3]

In the filter states, we model the thrust errors $m(t)$ by $R\epsilon(t)$, where R is a coordinate transformation matrix and $\epsilon(t)$ satisfies a first-order differential equation. The elements of $\epsilon(t)$ are the three orthogonal components of the acceleration error in the orbital reference frame. Any unknown parameters in the differential equations which describe $\epsilon(t)$ are also part of the state vector and are estimated along with position, velocity, and acceleration.

The filter dynamical model generally may be expressed as

$$\dot{x} = F(x,t) \qquad x(t_0) = x_0 \tag{9}$$

The specific equations in this vector are

$$\begin{aligned} \dot{r} &= v \\ \dot{v} &= -\frac{\mu}{|r|^3} r + T^* + R\epsilon \\ \dot{\epsilon} &= -\alpha\epsilon + U_\epsilon \\ \dot{\alpha} &= U_\alpha \end{aligned} \tag{10}$$

where α is a diagonal matrix and U_ϵ and U_α are random with the same statistics:

$$E[U] = 0 \qquad E[U(t)U(\tau)] = q(t)\delta(t-\tau)$$

where δ is the Kronecker delta.

The measurements used with this model are range rate from Earth and star tracker angles previously discussed.

As depicted in Fig. 3, the star trackers basically consist of optics that focus starlight from the navigation star onto a quad cell arrangement of photon counters. Since it is assumed that the vehicle will undergo only small angle motions about the star-tracker coordinate frame, the field of view is limited such that the navigation star is the dominant point light source. A threshold could be set to assure this fact. If the tracker is not pointing directly at the star because of off-nominal vehicle attitude, an Airy disk will be centered at a different position on the quad cell located at the focal plane of the optics. The imbalance of light intensity on the quad detector yields an imbalance of photoemissions from each side of the detector. The mean value for the rate of photoemissions for each part of the quad cell may be computed from the star intensity and the quantum efficiency of the detector. Because the photoemission is a Poisson process given the intensity rate, the counting events corresponding to a photoemission is a doubly stochastic Poisson process. The counts on each side of the quad cell are such that the mean value of the rate of photoemissions is directly proportional to the off-nominal pointing of the star trackers and thus proportional to vehicle attitude error – the source of thrust direction error. Since it is assumed that the thrust vector is fixed to the vehicle body, the sensed misalignment angles will be the same angles as the thrust misalignment relative to the orbital frame. Thus, the intensity rate models for each star tracker are functions of the misalignment angles and therefore are direct functions of the thrust direction. Hence, we express each star-tracker quad cell arrangement in terms of the intensity rates of photoemission.

Let η be the quantum efficiency, A the area of each quad of the quad cell, h Planck's constant, and ν the mean optical frequency. Then, from energy balance arguments and assuming small angles, the intensity rate on sides 1, 2, 3, and 4 of star tracker i may be written as

$$\lambda_1^i = \frac{\eta A I_0}{4h\bar{\nu}}\left(1 + C\theta_1^i + C\theta_2^i + C\theta_1^i\theta_2^i\right)$$

$$\lambda_2^i = \frac{\eta A I_0}{4h\bar{\nu}}\left(1 + C\theta_1^i - C\theta_2^i - C\theta_1^i\theta_2^i\right)$$

$$\lambda_3^i = \frac{\eta A I_0}{4h\bar{\nu}}\left(1 - C\theta_1^i + C\theta_2^i - C\theta_1^i\theta_2^i\right)$$

$$\lambda_4^i = \frac{\eta A I_0}{4h\bar{\nu}}\left(1 - C\theta_1^i - C\theta_2^i + C\theta_1^i\theta_2^i\right) \tag{11}$$

For small angles, the last term in each equation may be eliminated. The constant C represents a sensitivity factor which is a function of, among others, the optical focal length. It is assumed to be known a priori. These equations represent a mean value for the photoemission process in terms of misalignment angles, θ_1^i and θ_2^i, measured from the star tracker z axis, which nominally points at the star. These angles due to vehicle attitude motion may be written as

$$\theta_1^i = \sin^{-1}\left\{\frac{T_{ST_x}^i}{\sqrt{(T_{ST_y}^i)^2 + (T_{ST_x}^i)^2}}\right\} \tag{12}$$

$$\theta_2^i = \sin^{-1}\left\{\frac{T_{ST_y}^i}{\sqrt{(T_{ST_y}^i)^2 + (T_{ST_y}^i)^2}}\right\} \tag{13}$$

where T_{ST}^i is the heliocentric thrust vector expressed in the ith star tracker frame by

$$T_{ST}^i = R_1(\beta_i)R_2(\alpha_i)R_1(\gamma)R_2(\theta)R_3(\psi)T \tag{14}$$

where $R_3(\psi)$ is a transformation matrix representing a rotation about the third axis of a coordinate frame through an angle ψ; ψ, θ, and γ are shown in Fig. 1, α_i and β_i are shown in Fig. 2. Since total thrust, T, just used, consists of the nominal, T^*, plus the errors, $m(t)$, the star tracker photoemission counts directly contain information about thrust direction errors. These count measurements then, along with the range-rate measurement, comprise the basic measurement set. The next section gives the estimation equations for this problem.

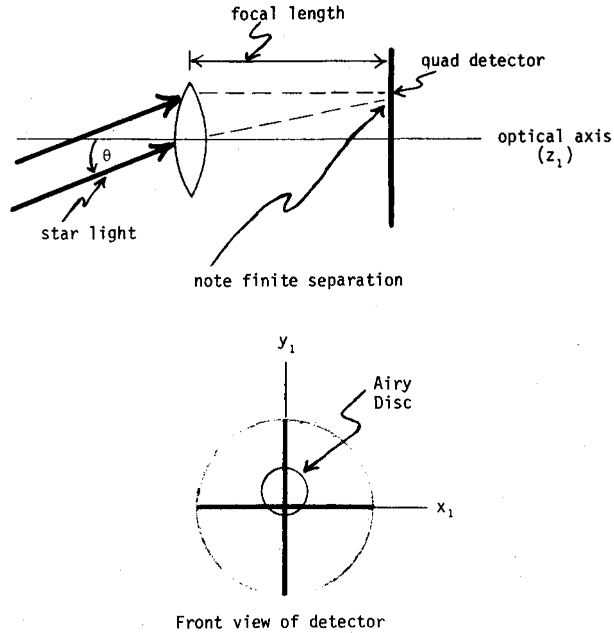

Fig. 3 Optical schematic of star tracker.

IV. Estimation Equations

This section contains the theory for the development of the estimation equations for the solar electric propulsion spacecraft problem. The first result is the differential equation for the measurement conditional probability density function. Next is a general equation for moment evolution. This is followed by the general conditional mean estimator and by an extended moment approximates to yield an implementable estimator.

Theorem 1: Differential Equation for the Conditional Probability Density Function with Mixed Observables

Let the system under consideration be that of Eq. (1) with the same assumptions and measurement subsystems. Then the partial integral differential equation for the conditional density function $P[x(t)|F_t]$ where F_t is the minimum σ algebra induced by the two measurement subsystems is given as

$$\begin{aligned} \mathrm{d}P[x(t)|F_t] &= LP[x(t)|F_t]\mathrm{d}t + [h[x(t),t] \\ &\quad - \hat{h}[x(t),t]]^T R(t)^{-1}[\mathrm{d}y(t) - \hat{h}[x(t),t]\mathrm{d}t]P[x(t)|F_t] \\ &\quad + \left\{\sum_{i=1}^{q}[\lambda_i[x(t),t] - \hat{\lambda}_i[x(t),t]]\hat{\lambda}_i[x(t),t]^{-1}\right. \\ &\quad \left.\cdot[\mathrm{d}N_i - \hat{\lambda}_i[x(t),t]\mathrm{d}t]P[x(t)|F_t]\right\} \end{aligned} \tag{15}$$

where

$$\hat{h}[x(t),t] = E^{F_t}[h[x(t,t]]$$

and

$$\hat{\lambda}[x(t),t] = E^{F_t}[\lambda[x(t),t]]$$

with

$$P[x(t_0)|F_0]=P[x(t_0)]$$

where L is the diffusion operator, i.e.,

$$L(\cdot)=-\sum_{i=1}^{n}\frac{\partial(\cdot f_i)}{\partial x_i}+\frac{1}{2}\sum_{i,j}^{n}\frac{\partial^2[\cdot(GQG^T)_{ij}]}{\partial x_i\,\partial x_y}$$

Proof

Appendix A contains the development of this theorem.

It may be seen that the differential equation for the conditional density function contains two measurement terms. The first corresponds to the case when only measurements with additive noise are available. The second corresponds to the case when only counting measurements are available. The reason for this additive feature is the conditional independence of the measurements. If other measurement subsystems with different types of measurements were added, then as long as they were conditionally independent, the measurement term as appears in the probability density differential equation when only one of the measurements was available, may be added to the diffusion operator to form the necessary differential equation.

Theorem 2: Evolution of Moments

Let $\phi[x(t)]$ be a twice continuously differentiable scalar function of the n vector x, which is the solution of Eq. (9). Then the differential equation for the F_t conditional expectation of ϕ, $E^{F_t}[\phi[x(t)]]$ is given as

$$\begin{aligned}
dE^{F_t}[\phi[x(t)]]&=E^{F_t}\{\phi_x[x(t)]^T f[x(t),t]\}dt\\
&+1/2\,E^{F_t}\{trG[x(t),t]Q(t)G[x(t),t]^T\phi_{xx}[x(t)]\}dt\\
&+\{E^{F_t}[\phi[x(t)]h[x(t),t]]-E^{F_t}[\phi[x(t)]]\\
&\cdot E^{F_t}[h[x(t),t]]\}^T R(t)^{-1}[dy(t)-E^{F_t}[h[x(t),t]]dt]\\
&+\sum_{i+1}^{q_\lambda}\{\{E^{F_t}[\phi[x(t)]\lambda_i[x(t),t]]-E^{F_t}[\phi[x(t)]]\\
&\cdot E^{F_t}[\lambda_i[x(t),t]]\}E^{F_t}[\lambda_i[x(t),t]]^{-1}\\
&\cdot[dN_i-E^{F_t}[\lambda_i[x(t),t]]dt]\}
\end{aligned}\tag{16}$$

where ϕ_x and ϕ_{xx} denote first and second derivatives of ϕ with respect to x.

Proof

The proof may be easily established by use of Theorem 1.

This theorem allows the development of all moments for the mixed observable problem. The next theorem yields the necessary estimation structure for the conditional mean estimator.

Theorem 3: Conditional Mean Estimator Structure

Given the system and measurement subsystems as previously defined, the equation for the conditional mean estimator is

$$\begin{aligned}
d\hat{x}(t)&=E^{F_t}[f[x(t),t]]dt+E^{F_t}\{[x(t)\\
&-\hat{x}(t)]h[x(t),t]^T\}R(t)^{-1}[dy(t)\\
&-E^{F_t}[h[x(t),t]]dt]\\
&+\sum_{i=1}^{q_\lambda}\{E^{F_t}[(x(t)-\hat{x}(t))\lambda_i[x(t),t]]\\
&\cdot E^{F_t}[\lambda_i[x(t),t]]^{-1}[dN_i-E^{F_t}[\lambda_i[x(t),t]]dt]\}
\end{aligned}\tag{17}$$

where $\hat{x}$ is the F_t-conditional mean and where the covariance of the estimation error is given as

$$\begin{aligned}
dP(t)&=\{E^{F_t}\{(x(t)-\hat{x}(t))f[x(t),t]^T\}\\
&+E^{F_t}\{f[x(t),t](x(t)-\hat{x}(t))^T\}\\
&+E^{F_t}\{G[x(t),t]Q(t)G[x(t),t]^T\}\}dt\\
&+E^{F_t}\{(x(t)-\hat{x}(t))(x(t)-\hat{x}(t))^T(h[x(t),t]\\
&-E^{F_t}[h[x(t),t]])^T\}[dy(t)-E^{F_t}[h[x(t),t]]dt]\\
&-E^{F_t}[(x(t)-\hat{x}(t))h[x(t),t]^T]R^{-1}E^{F_t}[h[x(t),t](x(t)\\
&-\hat{x}(t))^T]dt+\sum_{i=1}^{q_\lambda}E^{F_t}\}(x(t)-\hat{x}(t))(x(t)\\
&-\hat{x}(t))^T(\lambda_i[x(t),t]-E^{F_t}[\lambda_i[x(t),t]])\}\\
&\quad E^{F_t}[\lambda_i[x(t),t]]^{-1}[dN_i-E^{F_t}[\lambda_i[x(t),t]]dt]\\
&-\sum_{i=1}^{q_\lambda}E^{F_t}[(x(t)-\hat{x}(t))\lambda_i[x(t),t]]E^{F_t}[\lambda_i[x(t),t](x(t)\\
&-\hat{x}(t))^T]E^{F_t}[\lambda_i[x(t),t]]^{-2}dN_i
\end{aligned}\tag{18}$$

Proof

The proof is not shown as it may be easily established.

The following assumptions will be used to find an approximate estimation structure. First, it will be assumed that the estimate is to be unbiased. It will be assumed that fourth-order moments of the estimation error can be factored into products of second-order moments similar to Gaussian moments. It will be assumed that all remaining moments of the estimation error other than the second may be eliminated. With these assumptions, the approximate estimation structure may be written as

$$\begin{aligned}
d\hat{x}(t)&=f[\hat{x}(t),t]dt+P(t)\frac{\partial h}{\partial x}\Big|_{\hat{x}}R(t)^{-1}[dy(t)\\
&-h[\hat{x}(t),t]dt]+\sum_{i=1}\Big\{P(t)\frac{\partial\lambda_i}{\partial x}\Big|_{\hat{x}}\{\lambda_i[\hat{x}(t),t]\}^{-1}[dN_i\\
&-\lambda_i[\hat{x}(t),t]dt]\Big\}
\end{aligned}\tag{19}$$

and the approximate estimation error covariance is given as

$$\begin{aligned}
dP(t)&=\Big[P(t)\frac{\partial f}{\partial x}\Big|_{\hat{x}}+\frac{\partial f^T}{\partial x}\Big|_{\hat{x}}P(t)+G(t)Q(t)G(t)^T\\
&-\sum_i P(t)\frac{\partial^2\lambda_i}{\partial x^2}P(t)-P(t)\frac{\partial h^T}{\partial x}\Big|_{\hat{x}}R(t)^{-1}\frac{\partial h}{\partial x}\Big|_{\hat{x}}P(t)\Big]dt\\
&+\sum_i P(t)\frac{\partial^2\ell_n\lambda_i}{\partial x^2}\Big|_{\hat{x}}P(t)\lambda_i{}^T dN_i
\end{aligned}\tag{20}$$

where λ_i is a vector of zeros with a one in the ith row.

These results were first derived in Ref. 4-6 but by an elegant method using Martingales. Appendix A contains a much simpler and independent derivation.

The use of the state and measurement equations in Sec. III and the development of the extended filter in Eqs. (19) and (20) allow us to write the estimator as

$$\begin{bmatrix}\dot{\hat{r}}\\ \dot{\hat{v}}\\ \dot{\hat{\epsilon}}\\ \dot{\hat{\alpha}}\end{bmatrix}=\begin{bmatrix}\hat{v}\\ -\frac{\mu}{r^3}\Big|_{\hat{r}}\hat{r}+\tilde{R}\hat{\epsilon}\\ -\hat{\alpha}\hat{\epsilon}\\ 0\end{bmatrix}+\begin{bmatrix}0\\ T^*\\ 0\\ 0\end{bmatrix}$$

$$+P\frac{\partial h}{\partial x}\Big|_{\hat{r},\hat{v}} R^{-1}(\mathrm{d}\dot{\rho}-h(\hat{v})\mathrm{d}t)$$

$$+\sum_{i=1}^{8} P\frac{\partial \lambda_i}{\partial x}\Big|_{\hat{\epsilon}} \{\lambda_i(\hat{\epsilon})\}^{-1}[\mathrm{d}N_i-\lambda_i(\hat{\epsilon})\mathrm{d}t] \tag{21}$$

where $\partial h/\partial x$ and $\partial\lambda_i/\partial x$ are explained in Appendix B and $h(\hat{v})$ is the expected range-rate measurement evaluated at the current best estimate of v. $\lambda_i(\hat{\epsilon})$ is for $i=1$, 2, 3, and 4, the intensity rate for the first star tracker evaluated with Eqs. (11) and for $i=5$, 6, 7, and 8, the intensity rate for the second star tracker. Clearly, through the various coordinate transformations, the λ's will be a function of the thrust uncertainties $\hat{\epsilon}$ in heliocentric coordinates. The $\mathrm{d}\dot{\rho}$ and $\mathrm{d}N_i$'s are the differential measurements. That is, $\mathrm{d}\dot{\rho}$ is the differential in $\dot{\rho}$ in time $\mathrm{d}t$ and $\mathrm{d}N_i$ is the number of counts in time $\mathrm{d}t$ for the ith cell of the quad detectors.

The covariance $P(t)$ may be calculated by use of Eq. (20), where the partials are discussed in Appendix B.

We obtain

$$\mathrm{d}P=\Big[P\frac{\partial f}{\partial x}+\frac{\partial f^T}{\partial x}P+Q-\sum_{i=1}^{8}P\frac{\partial^2\lambda_i(\hat{\epsilon})}{\partial x^2}P$$

$$-P\frac{\partial h^T}{\partial x}\Big|_{\hat{x}} R^{-1}\frac{\partial h}{\partial x}\Big|_{\hat{x}} P\Big]\mathrm{d}t+\sum_{i=1}^{8}P\frac{\partial^2 \ln\lambda_i(\hat{\epsilon})}{\partial x^2}P\lambda_i^T\mathrm{d}N_i \tag{22}$$

where $\partial f/\partial x$ may be found in Ref. 1, the $\partial h/\partial x$ is in Appendix B, the second partials are discussed in Appendix B, and

$$Q=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q_\epsilon & 0 \\ 0 & 0 & 0 & q_\alpha \end{bmatrix}$$

It may be noted that the equation for the covariance of the estimation error is a function of the correct measurements. Thus, it is not possible to solve this equation offline. It is necessary to solve it with a particular measurement realization of detector measurements. This is a feature of the count observables not found in additive noise measurements.

V. Concluding Remarks

The equations developed previously and in the appendices have been set up to allow one to estimate the state of a randomly disturbed solar electric propulsion spacecraft using a mixture of two measurement types – those containing additive white noise and those based on a doubly stochastic Poisson process. It should be emphasized that the count measurements are coupled into the state error covariance equations.

Appendix A: Proof of Theorem 1

The proof of this theorem may be accomplished rigorously in a similar manner to Ref. 6. However, a simpler method of development similar to that in Kushner[7] will be used to establish the result. Proceeding, let

$$P[x(t)\mid Y_{t+\mathrm{d}t},N_{t_0,t+\mathrm{d}t}]=P[x(t)\mid Y_t,\mathrm{d}y_t,N_{t_0,t},\mathrm{d}N_t] \tag{A1}$$

where $Y_t=\{y(\sigma);\ t_p\leq\sigma\leq t\}$. The property that conditioned an $x(t)$, the incremental measurements $\mathrm{d}N_t$ and $\mathrm{d}Y_t$ are independent and Bayes rule may be used to establish that

$$P[x(t)\mid Y_{t+\mathrm{d}t},N_{t_0,t+\mathrm{d}t}]=\frac{P[\mathrm{d}N_t\mid x(t)]P[\mathrm{d}y_t\mid x(t)]P[x(t)\mid N_{t_0,t},Y_t]}{E^{F_t}\{P[\mathrm{d}N_t\mid x(t)]P[\mathrm{d}y_t\mid x(t)]\}} \tag{A2}$$

Define

$$\bar{R}[\mathrm{d}N_t,\mathrm{d}y_t,\mathrm{d}t,x(t)]=\frac{P[\mathrm{d}N_t\mid x(t)]P[\mathrm{d}y_t\mid x(t)]}{E^{F_t}\{P[\mathrm{d}N_t\mid x(t)]P[\mathrm{d}y_t\mid x(t)]\}} \tag{A3}$$

It is obvious that

$$P[\mathrm{d}y_t\mid x(t)]=C\exp\{-1/2\mathrm{d}t[\mathrm{d}y_t-h[x(t),t]\mathrm{d}t]^TR(t)^{-1}[\mathrm{d}y_t-h[x(t),t]\mathrm{d}t]\} \tag{A4}$$

and

$$P[\mathrm{d}N_t\mid x(t)]=\sum_{i=1}^{q\lambda}\lambda_i[x(t),t]\mathrm{d}N_i\mathrm{d}t+\Big[1-\sum_{i=1}^{q\lambda}\lambda_i[x(t),t]\mathrm{d}t\Big]\Big(1-\sum_{i=1}^{q\lambda}\mathrm{d}N_i\Big) \tag{A5}$$

(See Ref. 13.)

Then Eq. (A3) may be written as

$$\bar{R}[\mathrm{d}N_i,\mathrm{d}y_t,\mathrm{d}t,x(t)]=\frac{\Big[\sum_{i=1}^{q\lambda}\lambda_i\mathrm{d}N_i\mathrm{d}t+\Big(1-\sum_{i=1}^{q\lambda}\lambda_i\mathrm{d}t\Big)\Big(1-\sum_{i=1}^{q\lambda}\mathrm{d}N_i\Big)\Big]\exp\Big\{-\frac{1}{2\mathrm{d}t}(\mathrm{d}y_t-h\mathrm{d}t)^TR^{-1}(\mathrm{d}y_t-h\mathrm{d}t)\Big\}}{E^{F_t}\Big\{\Big[\sum_{i=1}^{q\lambda}\lambda_i\mathrm{d}N_i\mathrm{d}t+\Big(1-\sum_{i=1}^{q\lambda}\lambda_i\mathrm{d}t\Big)\Big(1-\sum_{i=1}^{q\lambda}\mathrm{d}N_i\Big)\Big]\exp\Big[-\frac{1}{2\mathrm{d}t}(\mathrm{d}y_t-h\mathrm{d}t)^TR^{-1}(\mathrm{d}y_t-h\mathrm{d}t)\Big]\Big\}} \tag{A6}$$

Equation (A6) may be written as

$$\bar{R}=\frac{\Big(1-\sum_{i=1}^{q\lambda}\lambda_i\mathrm{d}t\Big)T}{E^{F_t}\Big[\Big(1-\sum_{i=1}^{q\lambda}\lambda_i\mathrm{d}t\Big)T\Big]}\qquad\text{for } \mathrm{d}N_t=0 \tag{A7a}$$

$$\bar{R}=\frac{\lambda_j T}{E^{F_t}[\lambda_j T]} \quad \text{for } \mathrm{d}N_t=\gamma_j \tag{A7b}$$

where γ_j is a vector consisting of zeros except for the jth element which contains a one, and where

$$T=\exp\{1/2[2\mathrm{d}y^T R^{-1}h-h^T R^{-1}h\mathrm{d}t]\} \tag{A8}$$

Thus, $\overline{R}$ may be written as

$$\overline{R}=\frac{\left(1-\sum_{i=1}^{q\lambda}\lambda_i \mathrm{d}t\right)\left(1-\sum_{i=1}^{q\lambda}\mathrm{d}N_i\right)T}{E^{F_t}\left[\left(1-\sum_{i=1}^{q\lambda}\lambda_i \mathrm{d}t\right|T\right]}+\sum_{i=1}^{q\lambda}\frac{\lambda_j \mathrm{d}N_j T}{E[\lambda_j T]} \tag{A9}$$

This equation is equivalent to Eq. (A7) for the cases when $\mathrm{d}N$ equals zero or one. The limiting case is, in fact, that $\mathrm{d}N$ is equal to zero or one. For a similar procedure see Ref. 13.

Equation A3 may be expanded in a Taylor series about $\mathrm{d}y$ and $\mathrm{d}t$ equal to zero. This may be written as

$$\overline{R}\mathrm{d}N_t,\mathrm{d}y_t,\mathrm{d}t,x(t))=\overline{R}(\mathrm{d}N_t,0,0,x(t))+\frac{\partial\overline{R}}{\partial \mathrm{d}t}+\sum_{j=1}^{q}\frac{\partial\overline{R}}{\partial \mathrm{d}y_j}\mathrm{d}y_j+\frac{1}{2}\sum_{j,k}^{q}\frac{\partial^2\overline{R}}{\partial \mathrm{d}y_j\partial \mathrm{d}y_k}\mathrm{d}y_j\mathrm{d}y_k+\text{H.O.T.} \tag{A10}$$

The necessary evaluations may be easily established where they have been evaluated at the nominal values, i.e.,

$$\overline{R}[\mathrm{d}N_t,0,0,x(t)]=\left(1-\sum_{i=1}^{q\lambda}\mathrm{d}N_i\right)+\sum_{i=1}^{q\lambda}\lambda_i\hat{\lambda}_i^{-1}\mathrm{d}N_i \tag{A11}$$

$$\frac{\partial\overline{R}}{\partial \mathrm{d}t}=\left[\frac{1}{2}E^{F_t}(h^T R^{-1}h)+E^{F_t}\left(\sum_{i=1}^{q\lambda}\lambda_i\right)-\frac{1}{2}h^T R^{-1}h-\sum_{i=1}^{q\lambda}\lambda_i\right]\cdot\left(1-\sum_{i=1}^{q\lambda}\mathrm{d}N_i\right)+\sum_{i=1}^{q\lambda}\left\{\frac{\lambda_i\lambda_i^{-2}}{2}E^{F_t}(\lambda_i h^T R^{-1}h)-\frac{\lambda_i\lambda_i^{-1}}{2}h^T R^{-1}h\right\}\mathrm{d}N_i \tag{A12}$$

$$\frac{\partial\overline{R}}{\partial \mathrm{d}y_j}=\left[\sum_{\ell=1}^{q\lambda}r_{j_\ell}^{-1}h_\ell-E^{F_t}\left(\sum_{\ell-1}^{q}r_{j\ell}^{-1}h_\ell\right)\right]\left(1-\sum_{i=1}^{q}\mathrm{d}N_i\right)+\sum_{i=1}^{q\lambda}\left\{\lambda_i\hat{\lambda}_i^{-1}\sum_{k=1}^{q}r_{j_k}^{-1}h_k-\lambda_i\hat{\lambda}_i^{-2}E^{F_t}\left(\lambda_i\sum_{k=1}^{q}r_{j_k}^{-1}h_k\right)\right\}\mathrm{d}N_i \tag{A13}$$

and

$$\begin{aligned}\frac{\partial^2\overline{R}}{\partial \mathrm{d}y_j\partial \mathrm{d}y_k}=&\left\{\left(\sum_{\ell=1}^{q}r_{j_\ell}^{-1}h_\ell\right)\left(\sum_{\ell=1}^{q}r_{k_\ell}^{-1}h_\ell\right)-\left(\sum_{\ell=1}^{q}r_{1_\ell}^{-1}h_\ell\right)E^{F_t}\left(\sum_{\ell=1}^{q}r_{k_\ell}^{-1}h_\ell\right)\right.\\&-\left(\sum_{\ell=1}^{q}r_{k_\ell}^{-1}h_\ell\right)E^{F_t}\left(\sum_{\ell=1}^{q}r_{j_\ell}^{-1}h_\ell\right)-E^{F_t}\left[\left(\sum_{\ell=1}^{q}r_{j_\ell}^{-1}h_\ell\right)\left(\sum_{\ell=1}^{q}r_{k_\ell}^{-1}h_\ell\right)\right]\\&\left.+2E^{F_t}\left(\sum_{\ell=1}^{q}r_{j_\ell}^{-1}h_\ell\right)E^{F_t}\left(\sum_{\ell=1}^{q}r_{k_\ell}^{-1}h_\ell\right)\right\}\left(1-\sum_{i=1}^{q\lambda}\mathrm{d}N_i\right)+\sum_{i=1}^{q\lambda}\lambda_i\hat{\lambda}_i^{-1}\left(\sum_{\ell=1}^{q}r_{j_\ell}^{-1}h_\ell\right)\left(\sum_{\ell=1}^{q}r_{k_\ell}^{-1}h_\ell\right)\\&-\lambda_i\hat{\lambda}_i^{-2}\left(\sum_{\ell=1}^{q}r_{k_\ell}^{-1}h_\ell\right)E^{F_t}\left(\lambda_i\sum_{\ell=1}^{q}r_{j_\ell}^{-1}h_\ell\right)-\hat{\lambda}_1^{-2}\left[\lambda_i\left(\sum_{\ell=1}^{q}r_{j_\ell}^{-1}h_\ell\right)E^{F_t}\left(\lambda_i\sum_{\ell=1}^{q}r_{k_\ell}^{-1}h_\ell\right)\right.\\&\left.+\lambda_i E^{F_t}\left(\lambda_i\left(\sum_{\ell=1}^{q}r_{j_\ell}^{-1}h_\ell\right)\left(\sum_{\ell=1}^{q}r_{k_\ell}^{-1}h_\ell\right)\right)\right]+2\hat{\lambda}_i^{-3}\lambda_i E^{F_t}\left(\lambda_i\sum_{\ell=1}^{q}r_{j_\ell}^{-1}h_\ell\right)E^{F_t}\left(\lambda_i\sum_{\ell=1}^{q}r_{k_\ell}^{-1}h_\ell\right)\Big\}\mathrm{d}N_i\end{aligned} \tag{A14}$$

where $r_{j_\ell}^{-1}$ is the $j\ell$th element of the inverse matrix, R^{-1}. Therefore, one has that

$$P[x(t)\,|Y_{t+\mathrm{d}t},N_{t+\mathrm{d}t}]=(1+\lambda)P[x(t)\,|Y_t,N_t] \tag{A15}$$

where the definition of λ is obvious from Eqs. (A2) and (A10). Rewriting and taking the appropriate limits for the change due to both dynamics and measurements, since the change is obviously also due to dynamical changes, the appropriate differential equation is

$$\mathrm{d}P[x(t)\,|Y_t,N_t]=P[x(t)\,|Y_t,N_t]\mathrm{d}t+\lambda(t)P[x(t)\,|Y_t,N_t]$$

In this proof, the terms containing $\mathrm{d}N_i\mathrm{d}t$ may be eliminated by consideration of the change in the probability density at time $t_{\bar{E}}$ and a time t_E where t_E is an interarrival time for the counts. Terms containing $\mathrm{d}y_j\mathrm{d}N_i$ may be eliminated in a similar fashion.

Appendix B: Calculation of Partials

The range-rate measurement partials may be found in Ref. 1. They are:

$$\frac{\partial h}{\partial x}\bigg|_{\hat{x}}^{T}=\left[\frac{\partial\rho}{\partial r}\bigg|_{\hat{r},\hat{v}}^{T}\frac{\partial\rho^T}{\partial v}\bigg|_{\hat{r},\hat{v},}0\ 0\ 0\right]^T$$

where

$$\frac{\partial\dot{\rho}}{\partial r}=\begin{bmatrix}\frac{1}{\rho}\left[\dot{x}-\dot{x}_s-\frac{\dot{\rho}}{\rho}(x-x_s)\right]\\ \frac{1}{\rho}\left[\dot{y}-\dot{y}_s-\frac{\dot{\rho}}{\rho}(y-y_s)\right]\\ \frac{1}{\rho}\left[\dot{z}-\dot{z}_s-\frac{\dot{\rho}}{\rho}(z-z_s)\right]\end{bmatrix}$$

with $\dot{\rho}$ the range rate and ρ the range and

$$\frac{\partial \dot{\rho}}{\partial v} = \begin{bmatrix} (x-x_s)/\rho \\ (y-y_s)/\rho \\ (z-z_s)/\rho \end{bmatrix}$$

with x, y, z the heliocentric position and $\dot{x}$, $\dot{y}$, and $\dot{z}$ the velocity components of the vehicle, and $\dot{x}_s$, $\dot{y}_s$, and $\dot{z}_s$ are the heliocentric position and velocity components of the tracking station. The $\partial\lambda_i/\partial x$ vector will contain only terms $\partial\lambda_i/\partial\epsilon_j$ where $j=x,y,z$. In order to calculate this gradient, it is necessary to express the thrust vector in the star-tracker coordinate frame using the transformation defined in Eq. (14) and substitute the components into Eqs. (12) and (13). The θ then are functions of the ϵ_j's. Eliminating the cross terms in Eq. (11) and using the equation for the θ in terms of the ϵ yields equations for the intensity rate in terms of the ϵs. The partials may be then taken. Due to their complexity, they will not be shown here.

The second partials $\partial^2\lambda_i/\partial x^2$ and $\partial^2 \ln\lambda_i/\partial x^2$ may be similarly taken. It may prove easier to numerically compute the derivatives than to analytically compute them.

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